Scalar Conservation Laws with Random Initial Data

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Problem setting

We consider a scalar conservation law

$$\rho_t = H(\rho)_x,$$

with a random initial condition

$$\rho(0, x) = \xi(x).$$

Question

What can we say about the law of $\rho(t, \cdot)$?

Some ambiguity

Ambiguity

- Which probability space should the solution live in?
- What kind of probabilistic property we should consider?
- Under which sense we should solve the conservation law?

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Short answer

We are interested in the evolution of the distribution of the **entropy solution** with (spectrally negative) **Levy** initial data on the **canonical** probability space.

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Stochastic processes

Definition

We say a map $X : [0, \infty) \times \Omega \to \mathbf{R}$ is a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^X), \mathbb{P})$, and

- is $(\mathcal{F}_t^X)$-adapted if $X(t)$ is $\mathcal{F}_t^X$-measurable,
- is (further) Markov if $\mathbb{E}[X(t)|\mathcal{F}_s^X] \overset{\text{a.s.}}{=} \mathbb{E}[X(t)|X(s)]$ for any $t > s$,
- is (further) homogenous if $\mathbb{P}(X(t) \in \cdot|X(s))$ only depends on $t - s$.

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Canonical probability space

Let $D([0, \infty])$ denote the space of cadlag (right continuous and with left limit) paths. Almost all the stochastic processes of interest have a cadlag version modification, i.e., paths are cadlag $\mathbb{P}$-a.s. Thus, we can pushforward $(\Omega, \mathcal{F}, (\mathcal{F}_t^X), \mathbb{P}, X)$ to $(D, \mathcal{B}(D), (\mathcal{B}_t), X_{\#}\mathbb{P}, \bar{X})$.
Here, $\bar{X}$ is given by the evaluation map

$$\bar{X}(t, \omega) := \omega(t).$$

Levy processes

Definition

We say a stochastic process $X(t)$ is a Levy process on $(\Omega, \mathcal{F}, \mathbb{P})$ if it satisfies

- $X(0) = 0$ $\mathbb{P}$-a.s,
- $X(t) - X(s) \overset{\mathrm{d}}{=} X(t-s)$ for any $t > s$,
- $X(t) - X(s)$ is independent to $\mathcal{F}_s^X$ for any $t > s$,
- $\lim_{h \to 0} \mathbb{P}(|X(t+h) - X(t)| > \varepsilon) = 0$ for any t and ε.

We say U is an infinite divisible random variable if U can be written as a sum of n i.i.d. random variables for any n.

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Property

We point out for Levy process X, $X(1)$ is an infinite divisible random variable, and reversely given an infinite divisible random variable U there exists a unique (in law) Levy process X with $X(1) \stackrel{\mathrm{d}}{=} U$.

Examples (continuous)

We consider the most classical and popular example of continuous Levy process which is essentially a continuous limit of random walk.

Brownian motion

We say B_t is a Brownian motion if

- $B_0 = 0$ almost surely,
- The paths of B are almost surely continuous,
- B has independent increments,
- $B_t - B_s \overset{\mathrm{d}}{=} N(0, t-s)$.

Remark

The paths of B are actually $(\frac{1}{2} - \varepsilon)$-Holder continuous and have finite quadratic variation.

Examples (continuous)

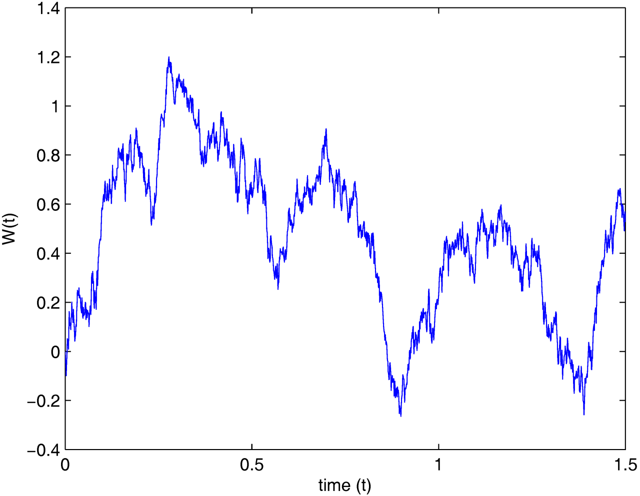

Figure: A sample path of the Brownian motion

Examples (continuous)

We give several characterizations of Brownian motion.

Definition

B_t is a Brownian motion if both B_t and $B_t^2 - t$ are continuous martingales starting at 0.

Construction

Let Z_n be a sequence i.i.d. normal random variables. Then,

$$B_t := \sum_n \sqrt{2} Z_n \frac{\sin((n - \frac{1}{2})\pi t)}{(n - \frac{1}{2})\pi}$$

is a Brownian motion.

Examples (jump)

Poisson process

We say N_t is a Poisson process with intensity $\lambda > 0$ if

- $N_0 = 0$ almost surely,
- The paths of N are almost surely cadlag,
- N has independent increments,
- $N_t - N_s \overset{\mathrm{d}}{=} \mathsf{Pois}(\lambda(t-s))$.

Compound Poisson process

Let D_n be a sequence of i.i.d. random variables. We say X_t is a compound Poisson process given by

$$X_t := \sum_{n=1}^{N_t} D_n.$$

Examples (jump)

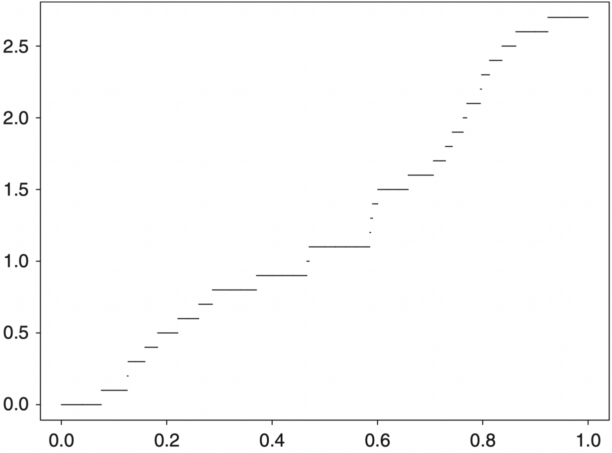

Figure: A sample path of the Poisson process

Characterization of Levy processes

Levy–Khinchine formula

Let X be a Levy process with characteristic exponent Ψ. Then, there exist (unique) $a \in \mathbf{R}, \sigma \geq 0$, and a measure Π, with no atom at zero, satisfying $\int (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$, such that

$$\Phi(\theta) = \frac{\log \mathbb{E}[\exp(i\theta X(t))]}{t} = ia\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbf{R}} [\mathrm{e}^{i\theta x} - 1 - i\theta x \mathbb{1}_{[-1,1]}(x)]\Pi(\mathrm{d}x)$$

Reversely, given a tuple (a, σ, Π) there exists a Levy process with the corresponding characteristic exponent.

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Reversely, given a tuple (a, σ, Π) there exists a Levy process with the corresponding characteristic exponent.

Levy–Ito decomposition

For Levy process X with the above characteristic exponent, it can be written as:

$$X_t = at + \sigma B_t + X_t^{(1)} + X_t^{(2)}.$$

$X^{(1)}$ and $X^{(2)}$ corresponds to the large jump and the small jump in X.

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Scalar conservation laws

Cauchy problem of a scalar conservation law is given by

$$\begin{cases} \rho_t = H(\rho)_x, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{1}$$

Distribution solution

Let $T > 0$ and denote $\pi_T = [0,T] \times \mathbf{R}$. Let $H : \mathbf{R} \to \mathbf{R}$ be C^1. We say a $\rho \in L^\infty$ is a distributional solution to (1) if it satisfies

$$\int_{\pi_T} \rho f_t - H(\rho) f_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbf{R}} \rho_0(x) f(0,x)\,\mathrm{d}x = 0,$$

for any test function $f \in C_c^\infty(\pi_T)$.
If u is a piecewise C^1 distributional solution, then it satisfies Rankine–Hugoniot condition

$$\dot{x}(t) = -\frac{H(\rho_-) - H(\rho_+)}{\rho_- - \rho_+}.$$

Scalar conservation laws

Entropy solution

We say $\rho : \pi_T \to \mathbf{R}$ is an entropy solution to (1) if it satisfies, for all $k \in \mathbf{R}$,

$$|\rho - k|_t - \operatorname{sgn}(\rho - k) H(\rho)_x \leq 0$$

in the distributional sense.

Entropy solution (alternative)

We say $\rho : \pi_T \to \mathbf{R}$ is an entropy solution to (1) if it satisfies, for all convex f with $g' = f' H'$,

$$f(\rho)_t - g(\rho)_x \leq 0$$

in the distributional sense.

Scalar conservation laws

Viscosity vanishing solution

Let $\rho^\varepsilon : \pi_T \to \mathbf{R}$ be the solution to

$$\rho^\varepsilon_t = H(\rho^\varepsilon)_x + \varepsilon \rho^\varepsilon_{xx}$$

with the initial condition

$$\rho^\varepsilon(0, x) = \rho_0(x).$$

We say $\rho : \pi_T \to \mathbf{R}$ is a viscosity vanishing solution to (1) if it is the limit of ρ^ε as $\varepsilon \to 0$.

Theorem

If ρ is a viscosity vanishing solution of (1), then ρ is an entropy solution.

Hamilton–Jacobi equation

The scalar conservation law (1) is closely related to the following Hamilton–Jacobi equation

$$\begin{cases} u_t = H(u_x), \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

Viscosity solution

We say u is a viscosity solution of (2), if it satisfies $u - \phi$ has a local maximum at point (t_0, x_0), then $\phi_t(t_0, x_0) \leq H(\phi_x(t_0, x_0))$ and $u - \psi$ has a local minimum at point (t_0, x_0), then $\psi_t(t_0, x_0) \geq H(\psi_x(t_0, x_0))$ for any $\phi, \psi \in C^\infty$.

Hamilton–Jacobi equation

Theorem

Let u be the unique viscosity solution of the Hamilton–Jacobi equation (2)*, then $\rho = u_x$ is the entropy solution of the scalar conservation law* (1) *with the initial condition*

$$\rho_0(x) = \frac{\mathrm{d}}{\mathrm{d}x} u_0(x).$$

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Let $H^*(s) = \sup_\rho(\rho s + H(\rho))$ denote the Legendre transform of $-H$ and call $u_0(s) = \int_0^s \rho_0(s)\,\mathrm{d}s$ the initial potential. We define the *Hopf–Lax functional*

$$I(s;x,t) = u_0(s) + tH^*(\frac{x-s}{t}).$$

The characteristic through (t,x) is given by the variational principle

$$y(t,x) = \sup\{s : I(s;x,t) = \inf_r I(r;x,t)\}.$$

We call $y(t,x)$ the inverse Lagrangian.

Hopf–Lax formula

Theorem

Assume H is strictly concave. The entropy solution of the scalar conservation law (1) is implicitly given by

$$H'(\rho(t,x)) = \frac{y(t,x) - x}{t}.$$

In particular, the entropy solution of the Burgers' equation $H(\rho) = -\rho^2/2$ has the form

$$\rho(t,x) = \frac{x - y(t,x)}{t},$$

where

$$y(t,x) = \arg^+ \max_s \{u_0(s) + \frac{(x-s)^2}{2t}\}.$$

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Conjecture I

In [Menon and Srinivasan(2010)] authors showed spectrally negative Markov process is preserved by the conservation law. Furthermore, they conjectured the evolution equation of the generator of the solution with **bounded variation spectrally negative Levy** initial data.
Let $\mathcal{A}(t)$ be the generator of $\rho(\cdot, t)$ which is given by

$$\mathcal{A}(t)J(y) = \lim_{x \to 0} \frac{\mathbb{E}^y[J(\rho(x,t))] - J(y)}{x},$$

for any $J \in C_c^\infty(\mathbf{R})$. For spectrally negative Levy process with bounded variation paths, its generator has the form of

$$\mathcal{A}(t)J(y) = b(t,y)J'(y) + \int_{-\infty}^{y} (J(z) - J(y))\, f(t, y, \mathrm{d}z)$$

where $b(t, y)$ characterizes the drift and $f(t, y, \cdot)$ describes the law of the jumps.

Conjecture II

The conjecture of [Menon and Srinivasan(2010)] is that the evolution of the generator $\mathcal{A}$ for $\rho(\cdot, t)$ is given by the Lax equation

$$\dot{\mathcal{A}} = [\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \tag{3}$$

for $\mathcal{B}$ which acts on test functions J by

$$\mathcal{B}(t)J(y) = -H'(y)b(t,y)J'(y) - \int_{-\infty}^{y} \frac{H(y) - H(z)}{y - z}(J(z) - J(y))\, f(t, y, \mathrm{d}z).$$

Literature review

Burgers' equation

- [Sinai(1992)] Brownian initial data
- [Avellaneda and Weinan(1995)] Tail distribution of white noise initial data
- [Bertoin(1998)] Brownian initial data
- [Carraro and Duchon(1998)] Statistical solution
- [Chabanol and Duchon(2004)] Statistical solution and the evolution equation

General scalar conservation laws

- [Menon and Srinivasan(2010)] Heuristic derivation of the evolution equation
- [Kaspar and Rezakhanlou(2016)] Pure jump Markov initial data
- [Kaspar and Rezakhanlou(2020)] Piecewise deterministic Markov initial data

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Kaspar and Rezakhanlou(2016)

Assumption (a)

The initial condition $\rho_0 = \rho_0(x)$ is a **bounded pure-jump Markov process** starting at $\rho_0(0) = 0$ and evolving for $x > 0$ according to a rate kernel $g(\rho_-, \mathrm{d}\rho_+)$. We assume that for some constant $P > 0$ the kernel g is supported on

$$\{(\rho_-, \rho_+) : 0 \leq \rho_- \leq \rho_+ \leq P\}$$

and has total rate which is constant in ρ_-:

$$\lambda = \int g(\rho_-, d\rho_+)$$

for all $0 \leq \rho_- \leq P$.

Kaspar and Rezakhanlou(2016)

Assumption (b)

The Hamiltonian function $H : [0, P] \to \mathbf{R}$ is smooth, convex, has nonnegative right-derivative at $p = 0$ and noninfinite left-derivative at $p = P$.

Evolution equation I

Definition

We say that a continuous mapping $f : [0, \infty) \to \mathcal{K}_+[0, P]$ is a *solution of the kinetic equation*

$$\begin{cases} f_t = \mathcal{L}^\kappa f \\ f(0, \rho_-, d\rho_+) = g(\rho_-, d\rho_+), \end{cases}$$

where

$$\begin{aligned} &\mathcal{L}^\kappa f(t, \rho_-, d\rho_+) \\ &= \int (H[\rho_*, \rho_+] - H[\rho_-, \rho_*]) f(t, \rho_-, d\rho_*) f(t, \rho_*, d\rho_+) \\ &\quad - \left[\int H[\rho_+, \rho_*] f(t, \rho_+, d\rho_*) - \int H[\rho_-, \rho_*] f(t, \rho_-, d\rho_*) \right] f(t, \rho_-, d\rho_+). \end{aligned}$$

Evolution equation II

Definition

We say that a continuous mapping $\ell : [0, \infty) \to \mathcal{M}_+[0, P]$ is a *solution of the marginal equation*

$$\begin{cases} \ell_t = \mathcal{L}^0 \ell \\ \ell(0, d\rho_0) = \delta_0(d\rho_0), \end{cases}$$

where

$$\mathcal{L}^0 \ell(t, d\rho_0) = \int H[\rho_*, \rho_0] \ell(t, d\rho_*) f(t, \rho_*, d\rho_0) - \left[\int H[\rho_0, \rho_*] f(t, \rho_0, d\rho_*) \right] \ell(t, d\rho_0).$$

Main theorem

Theorem

Under the above assumptions, the entropy solution ρ to

$$\begin{cases} \rho_t = H(\rho)_x \\ \rho(0,x) = \xi(x), \end{cases} \tag{4}$$

for each fixed $t > 0$ has $x = 0$ marginal given by $\ell(t, d\rho_0)$ and for $0 < x < \infty$ evolves according to rate kernel $f(t, \rho_-, d\rho_+)$.

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Remark

It has been shown in [Menon and Srinivasan(2010)] that the evolution equations for f and ℓ are equivalent to the Lax equation

$$\dot{\mathcal{A}} = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}.$$

Moreover, the well-posedness of the evolution equation is independent to this theorem.

Sketch of the proof

The proof of the main theorem can be decomposed into the following steps:

1. show the well-posedness of the evolution equation;
2. convert the problem to a problem of bounded area scalar conservation law with a random boundary condition;
3. construct a random particle system whose law corresponds to the evolution equation;
4. show the law of the bounded area problem is the same as the one induced by the random particle system.

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4. show the law of the bounded area problem is the same as the one induced by the random particle system.

Remark

The first step can be shown by classical discretization approximation. The second step comes from the finite speed of propagation. We will omit these two steps and focus on the last two steps due to the time constraint.

Bounded area problem

Theorem

For any fixed $L > 0$, consider the scalar conservation law

$$\begin{cases} \rho_t = H(\rho)_x & (x,t) \in (0,L) \times (0,\infty) \\ \rho = \xi & x \in [0,L] \times \{t = 0\} \\ \rho = \zeta & (x,t) \in \{x = L\} \times (0,\infty) \end{cases} \tag{5}$$

with initial condition ξ (restricted to $[0,L]$), open boundary at $x = 0$, and random boundary ζ at $x = L$. Suppose the process ζ has $\zeta(0) = \xi(L)$ and evolves according to the time-dependent rate kernel $H[\rho,\rho_+]f(t,\rho,d\rho_+)$ independently of ξ given $\xi(L)$. Then for all $t > 0$ the law of $\rho(\cdot,t)$ is as follows:

(i) *the $x = 0$ marginal is $\ell(t, d\rho_0)$, and*

(ii) *the rest of the path is a pure-jump process with rate kernel $f(t,\rho_-,d\rho_+)$.*

Random particle system

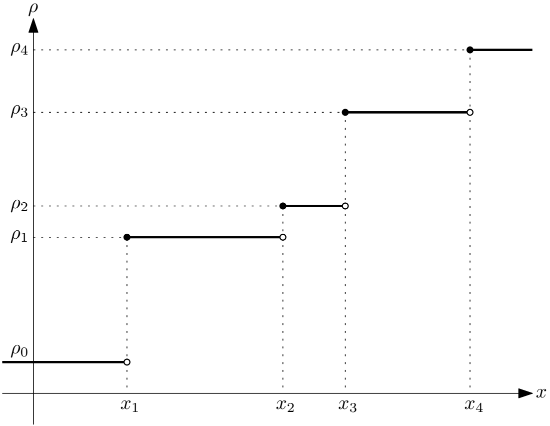

Figure: For each $t > 0$, the solution $\rho(x,t)$ is a nondecreasing, pure-jump process in x. We will see that for any fixed $L > 0$, we have a.s. finitely many jumps for $x \in [0, L]$ and that $\rho(\cdot, t)$ on this interval can be described by two (finite) nondecreasing sequences $(x_1, \ldots, x_N; \rho_0, \ldots, \rho_N)$.

Random particle system

Sticky dynamic description

$\rho(\cdot, t)$ can be fully characterized by the tuple $(x_1, \ldots, x_N; \rho_0, \ldots, \rho_N)$, where x_i are the position of shocks and ρ_i is the corresponding velocity. Now, we can view each shock as a particle, and its speed is decided by the Rankine–Hugoniot condition. To be more specific,

$$\dot{x}_i = -H[\rho_{i-1}, \rho_i] = -\frac{H(\rho_{i-1}) - H(\rho_i)}{\rho_{i-1} - \rho_i}.$$

The point here is that when two shocks collide, we should choose the correct solution, namely the entropy solution as follows. When $x_i = x_{i+1}$, the i^{th} particle is annihilated, and the velocity of the $(i+1)^{\text{th}}$ particle changes from $-H[\rho_i, \rho_{i+1}]$ to

$$\dot{x}_{i+1} = -H[\rho_{i-1}, \rho_{i+1}]. \tag{6}$$

In the case where several consecutive particles collide with each other at the same instant, all but the rightmost particle is annihilated.

Random particle system

For $C > 0$ and n a positive integer, write

$$\Delta_n^C = \{(a_1, \ldots, a_n) \in \mathbf{R}^n : 0 < a_1 < \cdots < a_n < C\}$$

and $\overline{\Delta_n^C}$ for the closure of this set in $\mathbf{R}^n$.

Configuration space

For L as in the bounded area problem, the configuration space Q for the sticky particle dynamics is

$$Q = \bigsqcup_{n=0}^{\infty} Q_n, \qquad Q_n = \Delta_n^L \times \overline{\Delta_{n+1}^P}.$$

A typical configuration is $q = (x_1, \ldots, x_n; \rho_0, \ldots, \rho_n) \in Q_n$ when $n > 0$, or $q = (\rho_0) \in Q_0 = \{\rho_0 : \ 0 \leq \rho_0 \leq P\}$ when $n = 0$.

Random dynamic system

Dynamics

Our notation for the particle dynamics is as follows:

(i) For $0 \leq s \leq t$ and $q \in Q$, write

$$\phi_s^t q = \phi_0^{t-s} q$$

according to the sticky particle dynamics, *without* random entry dynamics at $x = L$.

(ii) Given a configuration $q = (x_1, \ldots, x_n, \rho_0, \ldots, \rho_n)$ and $\rho_+ > \rho_n$, write $\epsilon_{\rho_+} q$ for the configuration $(x_1, \ldots, x_n, L, \rho_0, \ldots, \rho_n, \rho_+)$.

(iii) Write $\Phi_s^t q$ for the *random* evolution of the configuration with random boundary at $x = L$ according to the boundary process ζ. In particular, if the jumps of ζ between times s and t occur at times $s < \tau_1 < \cdots < \tau_k < t$ with values $\rho_{n+1} < \cdots < \rho_{n+k}$, then

$$\Phi_s^t q = \phi_{\tau_k}^t \epsilon_{\rho_{n+k}} \phi_{\tau_{k-1}}^{\tau_k} \epsilon_{\rho_{n+k-1}} \cdots \phi_{\tau_1}^{\tau_2} \epsilon_{\rho_{n+1}} \phi_s^{\tau_1} q.$$

Random dynamic systems

Remark

We remark that the described random dynamic system exactly characterizes the random dynamic of the bounded area problem. There is no need to let the number of the particles go to infinity to approximate the accurate dynamic. In fact, the number of the particles is also random in our particle systems.

Random measures

Our aim is to show that the law of $\Phi_0^t q$ has the desired marginal $\ell(t, d\rho_0)$ at $x = 0$ and the desired jump rate $f(t, \rho_-, d\rho_+)$.
We construct a candidate law $\mu(t, dq)$ on Q as follows. Take N to be Poisson with rate λL, $x_1, \ldots, x_N$ uniform on Δ_N^L, and $\rho_0, \ldots, \rho_N$ distributed on $\overline{\Delta_{N+1}^P}$ according to the marginal ℓ and transitions f independently of the x_i:

$$\mu(t, dq) := e^{-\lambda L} \sum_{n=0}^{\infty} \delta_n(dN) \mu_n(t, dq),$$

where $\mu_0(t, dq) = \ell(t, d\rho_0)$ and

$$\mu_n(t, dq) = \mathbb{1}_{\Delta_n^L}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n\, \ell(t, d\rho_0) \prod_{j=1}^{n} f(t, \rho_{j-1}, d\rho_j),$$

for $n > 0$.

Random measures

We then only need to show

$$\text{Law}(\Phi_0^t q) = \mu(t, dq) \tag{7}$$

where q has initial distribution $\mu(0, dq)$. Furthermore, we have the one-to-one map from Q to $\mathcal{M}[0, L]$ as

$$q \mapsto \pi(q, dx) = \rho_0 \delta_0 + \sum_{i=1}^{n} (\rho_i - \rho_{i-1}) \delta_{x_i}, \quad (q \in Q_n).$$

So, instead of (7) we would like to show the law of the random measure $\pi(\Phi_0^r q, \cdot)$ is identical to that of $\pi(q', \cdot)$ where q' is distributed by $\mu(t, dq')$.

Proof of step 4

Fix some time $T > 0$ and consider $F(t,q) = \mathbb{E}G(\Phi_t^T q)$ where G takes the form of a Laplace functional:

$$G(q) = \exp\left(-\int J(x)\,\pi(q,dx)\right) = \exp\left(-\rho_0 J(0) - \sum_{i=1}^{n}(\rho_i - \rho_{i-1})J(x_i)\right)$$

for $J \geq 0$ a continuous function on $[0,L]$. We aim to show that

$$\frac{d}{dt}\int \mathbb{E}G(\Phi_t^T q)\mu(t,dq) = 0 \tag{8}$$

for $0 < t < T$, from which it will follow that

$$\int \mathbb{E}G(\Phi_0^T q)\,\mu(0,dq) = \int G(q)\,\mu(T,dq)$$

and implies the result.

Formal calculation

We formally calculate

$$\int F(t,q)\mu(t,dq) - F(s,q)\mu(s,dq)$$
$$= \int (F(t,q) - F(s,q))\mu(t,dq) + \int F(s,q)(\mu(t,dq) - \mu(s,dq)).$$

Lemma

For any $n \geq 0$ and any $0 \leq s < t$ we have

$$\|\mu_n(t,\cdot) - \mu_n(s,\cdot) - (t-s)(\mathcal{L}^*\mu_n)(t,\cdot)\|_{TV} = o(t-s) \tag{9}$$

where the norm is total variation and $(\mathcal{L}^*\mu_n)(t,dq)$ is defined to be some signed kernel.

So, we only need to show

$$\int (F(t,q) - F(s,q))\mu(t,dq) \approx (t-s)\int F(s,q)\mathcal{L}^*\mu(s,dq).$$

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Bertoin(1998)

So far we only discussed the case that the solution has a finite variation. In [Bertoin(1998)], for Burges equation the Brownian initial data has been discussed. We recall the result here.

Theorem

Consider Burgers' equation with Brownian initial data $\xi(x)$. Then, for each fixed $t > 0$, the backward Lagrangian $y(x,t)$ has the property that

$$y(x,t) - y(0,t)$$

is independent of $y(0,t)$ and is a nondecreasing Levy process. Its distribution is the same as that of the first passage process

$$x \mapsto \inf\{z \geq 0 : t\xi(z) + z > x\}.$$

Statistical solution

Definition

A statistical solution of Burgers' equation is a sequence of probability measures $(\mu_t)_{t\geq 0}$ on $(D, \mathcal{B}(D))$ such that for any $v \in C_c^\infty$,

$$\partial_t \hat{\mu}_t(v) = i \int_D \int_{\mathbf{R}} \frac{1}{2} u(x)^2 v'(x)\, \mathrm{d}x \exp(i \int_{\mathbf{R}} u(x) v(x)\, \mathrm{d}x)\, \mathrm{d}\mu_t(\mathrm{d}u),$$

where $\hat{\mu}_t$ is the Fourier transform of μ_t given by

$$\hat{\mu}_t(v) = \int_D \exp(i \int_{\mathbf{R}} u(x) v(x)\, \mathrm{d}x)\, \mathrm{d}\mu_t(u).$$

Observation

From [Carraro and Duchon(1998)], we know Burgers' equation always has a statistical solution with a Levy initial data. However, only for the process without positive jump (rarefaction is excluded), its statistical solution coincides with the law of its entropy solution.

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Thank you!